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## MODIFIED FERMI ENERGY OF ELECTRONS IN A SUPERHIGH MAGNETIC FIELD

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In this paper, we investigate the electron Landau-level stability and its influence on the electron Fermi energy,  $E_F(e)$ , in the circumstance of magnetars, which are powered by magnetic field energy. In a magnetar, the Landau levels of degenerate and relativistic electrons are strongly quantized. A new quantity  $g_n$ , the electron Landau-level stability coefficient is introduced. According to the requirement that  $g_n$  decreases with increasing the magnetic field intensity  $B$ , the magnetic-field index  $\beta$  in the expression of  $E_F(e)$  must be positive. By introducing the Dirac- $\delta$  function, we deduce a general formulae for the Fermi energy of degenerate and relativistic electrons, and obtain a particular solution to  $E_F(e)$  in a superhigh magnetic field (SMF). This solution has a low magnetic-field index of  $\beta = 1/6$ , compared with the previous one, and works when  $\rho \geq 10^7 \text{ g cm}^{-3}$  and  $B_{\text{cr}} \ll B \leq 10^{17} \text{ Gauss}$ . By modifying the phase space of relativistic electrons, a SMF can enhance the electron number density  $n_e$ , and decrease the maximum of electron

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Landau level number, which results in a redistribution of electrons. According to Pauli exclusion principle, the degenerate electrons will fill quantum states from the lowest Landau level to the highest Landau level. As  $B$  increases, more and more electrons will occupy higher Landau levels, though  $g_n$  decreases with the Landau level number  $n$ . The enhanced  $n_e$  in a SMF means an increase in the electron Fermi energy and an increase in the electron degeneracy pressure. The results are expected to facilitate the study of the weak-interaction processes inside neutron stars and the magnetic-thermal evolution mechanism for magnetars.

*Keywords:* Neutron star; Equation of state; Fermi energy

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## 1. Introduction

It is universally recognized that pulsars are highly magnetized neutron stars (NSs), with surface dipole magnetic field being about  $10^{10} - 10^{12}$  Gauss. Magnetars are a kind of pulsars powered by their magnetic energy rather than their rotational energy, and their surface dipole magnetic fields are generally 2–3 orders of magnitude higher than those of radio pulsars ( $B^* = B/B_{\text{cr}} \gg 1$ ,  $B_{\text{cr}} = 4.414 \times 10^{13}$  Gauss is the quantum critical field of electrons), and their internal magnetic fields might be even higher (e.g., see Ref.1). Magnetars are categorized into two populations historically: Soft Gamma-ray Repeaters (SGRs) and Anomalous X-ray Pulsars (AXPs). The SGR flares were explained as resulting from violent magnetic reconnections and crustal quakes, and the quiescent X-ray emission of AXPs (with X-ray luminosities much larger than their spin-down luminosities) was attributed to the decay of superhigh magnetic fields (e.g., see Ref. 2, 3) under which the Landau levels of electrons are strongly quantized.

For the completely degenerate and relativistic electrons in  $\beta$ -equilibrium, the distribution function  $f(E_e)$  obeys Fermi-Dirac statistics (see Ref. 4). When the temperature  $T \rightarrow 0$ , the electron chemical potential  $\mu_e$  is also called “the electron Fermi energy”,  $E_F(e)$ , which has the simple form of  $E_F^2(e) = p_F^2(e)c^2 + m_e^2c^4$ , with  $p_F(e)$  being the electron Fermi momentum. As an extremely important and indispensable physical parameter in the equation of state (EoS) of a NS, the Fermi energy of electrons directly exerts impact on the weak-interactions processes, including modified Urca reactions,  $\beta$ -decay, electron capture, as well as the absorption of neutrinos and anti-neutrinos, etc (see Ref. 5, 6). They will in turn influence the intrinsic EoS, internal structure, thermal evolution, and even the overall properties of the star (see Ref. 49). Therefore, it is of great significance to study  $E_F(e)$  in the circumstance of a NS.

Since  $E_F(e)$  increases with the increasing in the depth of a NS (see Ref. 8), it is necessary to briefly review the structure of the star. The structure of a NS roughly includes an atmosphere and four major internal regions: the outer crust, inner crust, outer core, and inner core. The outer shell comprises crystal lattices and electrons, which are distributed from the surface of the star to the region where the neutron-drop density (see Ref. 9)  $\rho_d$  is reached. At the point  $\rho = \rho_d$ ,

the neutrons begin to overflow from the nuclei, forming a free neutron gas, and the value of  $E_F(e)$  is about 25 MeV (see Ref. 9, 10). The inner shell is mainly composed of degenerate and relativistic electrons, non-relativistic nucleons and over-saturated neutrons, distributed from the region of neutron-drop density to the shell-core boundary ( $\rho \sim 0.5\rho_0$ ,  $\rho_0 = 2.8 \times 10^{14} \text{ g cm}^{-3}$  is the standard nuclear density). Nuclei fully disappear at this density, and  $E_F(e)$  is about 35 MeV. The outer core is composed of neutrons and a small amount of electrons and protons, with the density range being  $0.5\rho_0 \sim 2.5\rho_0$ . The Fermi energy of electrons in this region is estimated as  $E_F(e) = 60 \times (\rho/\rho_0)^{2/3} \text{ MeV}$  (see Ref. 8). For the region with higher density, the electron Fermi energy could exceed the muon rest-mass energy  $m_\mu c^2 = 105.7 \text{ MeV}$ , and a small amount of muons ( $\mu$ ) are detected. The inner core is about several kilometers in radius, and has a central density as high as  $\sim 10^{15} \text{ g cm}^{-3}$ . When  $\rho > \rho_{tr}$ , some nucleons will transform to exotic particles such as hyperons, pion condensates, kaon condensates, quarks and etc. Here  $\rho_{tr}$  is the transition density of singular particles. To date, the value of  $\rho_{tr}$  is uncertain. For example, Tsuruta et al.(2009) (see Ref. 11) gave an estimate  $\rho_{tr} \sim 4\rho_0$ .

What we are most interested in is how a SMF can influence Landau levels of degenerate and relativistic electrons and their Fermi energy. Many authors (see Ref. 12, 13, 14, 15) have carried out detailed studies on the influences of a SMF on the composition and the EOS of a NS. According to the requirement of quantization of Landau levels, we introduced the Dirac- $\delta$  function (see Ref. 16, 17), and obtained a particular solution to  $E_F(e)$ ,

$$E_F(e) \simeq 43.44 \times \left( \frac{\rho}{\rho_0} \frac{Y_e}{0.0535} \frac{B}{B_{cr}} \right)^{1/4} \text{ MeV} \quad (B \gg B_{cr}), \quad (1)$$

where  $Y_e$  is the electron fraction, which is defined as  $Y_e = n_e/n_B$ , where  $n_e$  and  $n_B$  are the electron number density, and the baryon number density, respectively (see Ref. 16, 17). Furthermore, we deduced a general expression for  $P_e$ , the pressure of relativistic electrons (see Ref. 4), discussed the quantization of the electron Landau levels, and explored the influence of quantum electrodynamics effects on the EoS. The main conclusions included: The higher the magnetic field intensity, the bigger the electron pressure, and the high pressure is caused by high Fermi energy of electrons; the total pressure of a magnetar is always anisotropic; compared with an ordinary radio pulsar, a magnetar might be a denser NS if the anisotropic total pressure is taken into consideration; a magnetar might have larger mass if the positive energy contribution of the magnetic field energy to the EoS is taken into consideration (see Ref. 4).

Our research results pose a challenge to the prevalent viewpoint (see Ref. 18, 19): In a SMF, the higher the magnetic field intensity  $B$ , the lower the electron Fermi energy and the electron pressure. This prevalent viewpoint essentially goes against the real requirement of the quantization of Landau levels, due to the introduction of an artificial and false assumption and the application of the solution of a non-relativistic electron cyclotron motion equation (see Ref. 4) for specific information).

Recently, after a careful examination, we found some inadequacies of our theoretical model, mainly including the following aspects: 1) No consideration was given to the stability of Landau levels of electrons in a strong magnetic field. Till now, there has yet been no any relevant works or explicit analytical expression on the stability coefficient  $g_n$  in the physics community due to the complexity of this issue; 2) No explicit analytic expression for  $E_F(e)$  and  $n_e$  was provided. There was no comparison between the relationship of  $E_F(e)$  and  $n_e$  in a SMF with that in a weak magnetic field approximation ( $B^* \ll 1$ ), based on which, the variation range of the magnetic field index  $\beta$  in the expression is defined; 3) In the expression of  $E_F(e)$ , the application scope of  $B$  was not clearly defined because the Fermi surfaces (in the momentum space) of electrons in a non-relativistic magnetic field are basically spherically symmetrical, whereas the Fermi ball (in the momentum space) is turned into Landau cylinder in a relativistic magnetic field (see Ref. 20, 21, 22); 4) The most important thing is that the physical meaning of the magnetic field index  $\beta$  ( $\beta = 1/4$ ) in the expression of  $E_F(e)$  (see Ref. 16, 17) is not clear.

With the increase in  $B$ , the Landau cylinder becomes longer and narrower. When the SMF is too high, the Landau cylindrical space will be streamlined into a one-dimensional linear chain, making our model no longer applicable. Simply speaking, much detailed information in our previous works has been neglected (especially, ignoring the discrepancy of different Landau levels of electrons) in the process of derivation of  $E_F(e)$  and/or  $P_e$ . Therefore, it is of great importance to modify the expression  $E_F(e)$  in a SMF.

This paper is organized as follows: In Section 2, we review the relationship between  $E_F(e)$  and  $n_e$  in the weak magnetic field approximation; in Section 3, we deduce a general expression of  $E_F(e)$  in a SMF by introducing the stability coefficient of the Landau levels, and modify the particular solution to  $E_F(e)$ , and in Section 4, we present our summary and discussion.

## 2. The Fermi Energy in The Weak Magnetic Field Approximation

This part mainly refers to Ref. 49. Based on the basic definition of the Fermi energy of relativistic electrons, we obtain a particular solution to  $E_F(e)$ ,

$$E_F(e) = 60 \times \left(\frac{\rho}{\rho_0}\right)^{1/3} \left(\frac{Y_e}{0.005647}\right)^{1/3} \quad (\text{MeV}). \quad (2)$$

which is suitable for relativistic electron matter region in a NS. By means of numerical simulation, we obtained some analytic expressions of  $Y_e$  and  $\rho$  for several reliable EoSs with which we can estimate  $E_F(e)$  at any matter density by combining these analytical expressions with boundary conditions<sup>49</sup>.

Although  $E_F(e)$  in a weak magnetic field approximation could be expressed as the function of  $Y_e$  and  $\rho$ , the Fermi energy of electrons is solely determined by the electron number density  $n_e$ . Since electrons are extremely relativistic, the dimensionless electron Fermi momentum  $x_e = p_F(e)/m_e c \gg 1$ , then we obtained

the relationship between  $E_F(e)$  and  $n_e$ ,

$$\begin{aligned} E_F(e) &= m_e c^2 (1 + x_e^2)^{1/2} \approx m_e c^2 x_e \\ &= m_e c^2 (n_e 3\pi^2 \lambda_e^3)^{1/3} \\ &= \hbar c (3\pi^2 n_e)^{1/3} = 6.12 \times 10^{-11} n_e^{1/3} \text{ (MeV)}, \end{aligned} \quad (3)$$

in a weak magnetic field approximation, where  $\lambda_e = h/m_e c = 2.4263 \times 10^{-10}$  cm is the electron Compton wavelength.

All the other Fermi parameters of electrons are also solely determined by the number density of free electron gas,  $n_e$ . For example, the electron Fermi momentum  $p_F(e) = \hbar k_F = \hbar (3\pi^2 n_e)^{1/3}$ , where  $k_F = (3\pi^2 n_e)^{1/3}$  is the Fermi wave vector of electrons. For the Fermi kinetic energy of relativistic electrons,  $E_K^F(e) \approx c p_F(e) = \hbar c (3\pi^2 n_e)^{1/3}$ , and  $E_K^F(e) \gg m_e c^2$ . However, the relations between  $n_e$  and  $\rho$  in different density regions of a NS are usually unknown, and the known relations of  $n_e$  and  $\rho$  depend on the EoS in some specific matter models, and on the analytical expression of  $Y_e$  and  $\rho$  obtained from the EOS in a certain matter model.

### 3. Electron Fermi Energy in a Superhigh Magnetic Field

#### 3.1. Stability of Electron Landau Level

We now consider a uniform magnetic field  $B$  directed along the z-axis. In this case, in the Landau gauge the vector potential  $\vec{A}$  reads  $\vec{A} = (-B_y, 0, 0)$ . For extremely strong magnetic fields, the cyclotron energy becomes comparable to the electron rest-mass energy, and the transverse motion of the electron becomes relativistic. A relativistic magnetic field is often called the quantum critical magnetic field ( $B_{cr} = m_e^2 c^3 / e \hbar = 4.414 \times 10^{13}$  Gauss), which is obtained from the relation  $\hbar \omega = m_e c^2$ . The electron energy levels may be obtained by solving the relativistic Dirac equation in a strong magnetic field with the result

$$E_e^2 = m_e^2 c^4 (1 + 2\nu \frac{B}{B_{cr}}) + p_z^2(e) c^2, \quad (4)$$

where the quantum number  $\nu$  is given by  $\nu = n + 1/2 + \sigma'$ , the Landau level number  $n = 0, 1, 2, \dots$ , spin  $\sigma' = \pm 1/2$ , and the quantity  $p_z(e)$  is the z-component of the electron momentum, deemed as a continuous function. In a SMF, the maximum z-momentum of electrons  $p_z^F(e)$  is defined (see Ref. 18) by

$$(p_z^F(n)c)^2 + m_e^2 c^4 + (2n + 1 + \sigma) m_e^2 c^4 B^* \equiv E_F^2(e), \quad (5)$$

where the range of  $p_z^F(e)$  is  $0 \sim E_F(e)/c$ . For given values of the magnetic field intensity, the Fermi energy and the z-momentum of electrons, the electron Landau level number  $n$  is given by

$$n(\sigma = -1) = \text{Int} \left[ \frac{1}{2B^*} \left[ \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_z^F(e)}{m_e c} \right)^2 \right] \right], \quad (6)$$

$$n(\sigma = 1) = \text{Int} \left[ \frac{1}{2B^*} \left[ \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_z^F(e)}{m_e c} \right)^2 \right] - 1 \right], \quad (7)$$

where  $\text{Int}[x]$  denotes an integer value of the argument  $x$ , and  $\sigma = 2\sigma' = \pm 1$  is the spin projection value.

In a weak magnetic field  $B^* \ll 1$ , for the electron gas in the non-degenerate limit (temperature different from zero), the maximum Landau level number,  $n_m \rightarrow \infty$ . However, the maximum Landau level number  $n_m$  is set by the condition  $p_z^F(e) \geq 0$  or

$$E_F^2(e) \geq m_e^2 c^4 \left( 1 + 2\nu \frac{B}{B_{cr}} \right), \quad (8)$$

for highly degenerate electron gas in a SMF (see Ref. 18). It is obvious that the maximum of the electron Landau level number decreases with  $B$  when  $E_F(e)$  and  $p_z^F(e)$  are given. This is because the higher the magnetic field intensity, the more unstable the Landau levels of electrons, and the bigger the Landau level number  $n$ , the lower the Landau-level stability.

Indeed, the issue concerning the Landau-level stability of charged particles in a SMF is so complicated that there has not been any relevant work or explicit analytical expression in the physics community. In our previous works (see Ref. 16, 17), we did not take into consideration the Landau-level stability of electrons in a SMF that limits the application of our model. In this work, a new quantity,  $g_n$ , the Landau-level stability coefficient of electrons in a SMF, is introduced. Considering the uncertainty of the electron microscopic states in a SMF, we assume that  $g_n$  takes the power-law form

$$g_n = g_0 n^\alpha \quad (n \geq 1), \quad (9)$$

where  $n$ ,  $g_0$  and  $\alpha$  are the Landau level number, the ground-state level stability coefficient, and the stability index of Landau levels, respectively. When  $n = 1$ ,  $g_1 = g_0$ , i.e., the ground-state level has the same stability as that of the first excited level. According to quantum mechanics, the electrons at a higher energy level are prone to have excited transitions towards a lower energy level. The bigger the Landau level number, the shorter the level-occupying time for electrons, and the lower the Landau-level stability, the higher the probability of the excited transition.

Since the ground state level has the highest stability and  $g_n$  decreases with increasing  $n$ , i.e.,  $g_n < g_{n-1} < g_{n-2}$ , the stability index  $\alpha$  should be negative. The main reasons are as follows: If  $\alpha = 0$ , then  $g_n = g_{n-1} = \dots = g_1 = g_0$ , i.e., all the Landau levels have the same stability, and the maximum of the Landau level number,  $n_m$ , can take any high value. This scenario is essentially corresponding to a weak magnetic field approximation; if  $\alpha > 0$ , then  $g_n > g_{n-1} > \dots = g_1 = g_0$ , and under such a condition, a higher Landau level number means a higher stability, and  $n$  can also take any high value, which is clearly contrary to the principles of quantum mechanics. According to the analysis above, for degenerate and relativistic

electrons in a SMF, the Landau-level stability index,  $\alpha < 0$ . Based on Eq.(9), we make a schematic diagram of  $g_n$  and  $\alpha$ , shown in Fig. 1.

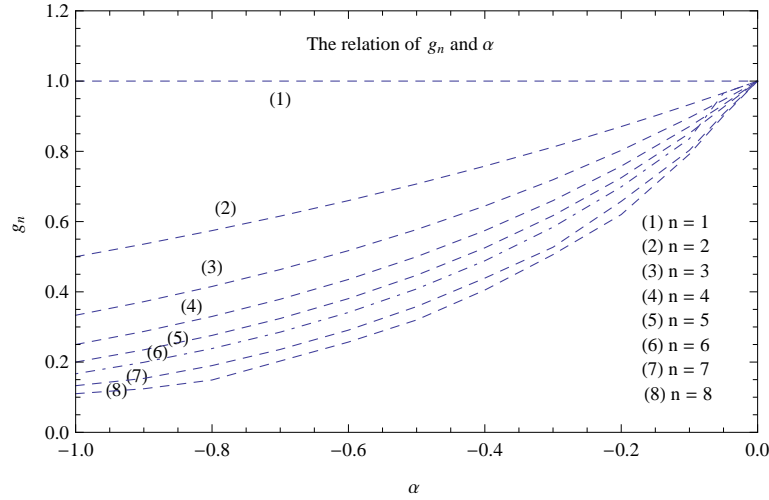


Fig. 1. The diagrams of  $g_n$  vs.  $\alpha$  for electrons in a superhigh magnetic field.

As seen from Fig.1, for a given Landau level with  $n \geq 1$ , the coefficient  $g_n$  increases with  $\alpha$  slowly, and the bigger the Landau level number  $n$ , the faster the change of  $g_n$  with  $\alpha$ . From discussions above, the bigger the Landau level number  $n$ , the greater the influence of the stability index  $\alpha$  on  $g_n$ , and the larger the probability of a particle's transition from a higher energy level into a lower energy level.

It should be pointed out that, in atomic physics and statistics mechanics (see Ref. 23, 25) the statistical weight describes the energy state density of microscopic particles, in other words, a higher energy level number means a bigger statistical weight. The higher the quantum number  $\nu$ , the wider the energy level width, and the more the microscopic state number of particles is. Here the Landau-level stability coefficient for electrons and the statistical weight are two totally different concepts.

### 3.2. The Energy State Density of Electrons in Phase Space

In a SMF, the Fermi surface of electrons becomes a narrow Landau cylinder. Combining  $B_{cr} = m_e^2 c^3 / e \hbar$  with  $\mu_e = e \hbar / 2 m_e c$ , Eq.(4) is modified as

$$E_e^2 = m_e^2 c^4 + p_z^2(e) c^2 + (2n + 1 + \sigma) 2 m_e c^2 \mu_e B, \quad (10)$$

where  $\mu_e \sim 0.927 \times 10^{-20}$  ergs Gauss $^{-1}$  is the magnetic moment of an electron. The continuous physical variables  $p_x$  and  $p_y$  (see Ref. 20), adopted in a non-relativistic magnetic field, will be no longer applicable. Thus, a quantized or discrete relativistic variable  $p_{\perp}$  must be adopted for replacement, where  $p_{\perp}$  is the electron

momentum perpendicular to the magnetic field,  $p_{\perp} = m_e c ((2n + 1 + \sigma) B^*)^{1/2}$ . The microscopic state number in a 6-dimension phase-space element  $dx dy dz dp_x dp_y dp_z$  is  $dx dy dz dp_x dp_y dp_z / h^3$ . By using the relation  $2\mu_e B_{cr} / m_e c^2 = 1$  and summing over the electron energy states in a 6-dimension phase space, we can express the electron energy state density  $N_{pha}$  as follows

$$N_{pha} = \frac{2\pi}{h^3} \int dp_z \sum_{n=0}^{n_m(\sigma, B^*)} \sum g_n \times \int \delta\left(\frac{p_{\perp}}{m_e c} - [(2n + 1 + \sigma) B^*]^{1/2}\right) p_{\perp} dp_{\perp}. \quad (11)$$

where  $\delta(p_{\perp}/m_e c - [(2n + 1 + \sigma) B^*]^{1/2})$  is the Dirac- $\delta$  function for electrons in a SMF. The physical significance of the Dirac- $\delta$  function lies in that, there doesn't exist any microscopic quantum state between the  $n$ -th and  $n + 1$ -th Landau torus due to the strong quantization of the electron Landau levels. For the ground state level, the electron spin is antiparallel to  $B$ , so the Landau level is non-degenerate ( $n = 0, \sigma = -1$ ); whereas higher levels are doubly degenerate ( $n \geq 1, \sigma = \pm 1$ ). Thus Eq.(11) can be rewritten as

$$N_{pha} = 2\pi \left(\frac{m_e c}{h}\right)^3 \int_0^{\frac{p_z^F(e)}{m_e c}} d\left(\frac{p_z}{m_e c}\right) g_n \times \left[ \sum_{n=0}^{n_m(B^*, \sigma=-1)} \int \delta\left(\frac{p_{\perp}}{m_e c} - (2n B^*)^{1/2}\right) \left(\frac{p_{\perp}}{m_e c}\right) d\left(\frac{p_{\perp}}{m_e c}\right) + \sum_{n=1}^{n_m(B^*, \sigma=1)} \int \delta\left(\frac{p_{\perp}}{m_e c} - (2(n+1) B^*)^{1/2}\right) \left(\frac{p_{\perp}}{m_e c}\right) d\left(\frac{p_{\perp}}{m_e c}\right) \right], \quad (12)$$

The upper limit of summation on Eq.(12) is  $n_m(B^*)$ , which has the following approximate relation,

$$n_m(B^*, \sigma = 1) \simeq n_m(B^*, \sigma = -1) = n_m(B^*) \approx \text{Int} \left[ \frac{1}{2B^*} \times \left( \frac{E_F(e)}{m_e c^2} \right)^2 \right]. \quad (13)$$

when  $n_m(B^*) \gg 1$ . In deriving the above expression, we have taken into account the dependence of  $n_m(B^*)$  on  $p_z^F(e) \geq 0$ , and assumed  $\frac{E_F(e)}{m_e c^2} \gg 1$ , and the lowest limit  $p_z^F(e) \rightarrow 0$ . Inserting Eq.(13) into Eq.(12) yields

$$N_{pha} = 2\pi \left(\frac{m_e c}{h}\right)^3 g_0 \int_0^{\frac{p_z^F(e)}{m_e c}} \sqrt{2B^*} \times \sum_{n=0}^{n_m(B^*)} n^{\alpha} (\sqrt{n} + \sqrt{n+1}) d\left(\frac{p_z}{m_e c}\right). \quad (14)$$



With a more rigorous replacement of integral upper limit  $\int_0^{\frac{p_z^F(e)}{m_e c}} \rightarrow \int_0^{\frac{E_F(e)}{m_e c^2}}$  (the range of  $p_z^F(e)$  is  $0 \sim E_F(e)/c$ ), Eq.(14) can be simplified as,

$$N_{pha} = 2^{\frac{5}{2}} \pi \sqrt{B^*} \left(\frac{m_e c}{h}\right)^3 g_0 \int_0^{\frac{E_F(e)}{m_e c^2}} \sum_{n=0}^{n_m(B^*)} n^{\alpha+1/2} d\left(\frac{p_z}{m_e c}\right). \quad (15)$$

Note that if the matter density is so high that the electron longitudinal kinetic energy exceeds its rest-mass energy, or if the magnetic field is so high that the electron cyclotron energy exceeds its rest-mass energy, then the electron becomes relativistic. Here we introduce a ratio  $q(\alpha)$ , which is defined as  $q(\alpha) = I_1/I_2$ ,  $I_1 = \int_0^{n_m(B^*)} n^{\alpha+1/2} dn$  and  $I_2 = \sum_{n=0}^{n_m(B^*)} n^{\alpha+1/2}$ . For a given index  $\alpha$  ( $\alpha < 0$ ), assuming  $n_m(B^*)$  to be 6, 8, 10, 15, 20 and 30 at random, we can calculate the corresponding values of  $q(\alpha)$ , as listed in Table 1.

Table 1. Values of  $q(\alpha)$  assuming different  $\alpha$  and  $n_m(B^*)$ .

$n_m(B^*)$	$q(\alpha)$ ( $\alpha = -0.1$ )	$q(\alpha)$ ( $\alpha = -0.5$ )	$q(\alpha)$ ( $\alpha = -0.8$ )	$q(\alpha)$ ( $\alpha = -0.95$ )
6	0.83	0.86	0.88	0.90
8	0.87	0.89	0.91	0.93
10	0.89	0.91	0.93	0.94
15	0.92	0.94	0.95	0.96
20	0.94	0.95	0.97	0.97
25	0.95	0.96	0.97	0.97
30	0.96	0.97	0.97	0.98

From Table 1, it is easy to see that  $q(\alpha)$  increases with  $n_m(B^*)$ , and  $q \simeq 1$  if  $n \gg 1$ . Thus, the following summation formula is approximately replaced by an integral equation,

$$\sum_{n=0}^{n_m(B^*)} n^{\alpha+\frac{1}{2}} \simeq \int_0^{n_m(B^*)} n^{\alpha+\frac{1}{2}} dn = \frac{2}{2\alpha+3} n_m^{\alpha+\frac{3}{2}}, \quad (16)$$

when  $n_m(B^*) \geq 6$ . Then Eq.(15) can be rewritten as

$$N_{pha} = \frac{2^{\frac{7}{2}}}{2\alpha+3} \pi \sqrt{B^*} \left(\frac{m_e c}{h}\right)^3 g_0 \int_0^{\frac{E_F(e)}{m_e c^2}} \left[ \frac{1}{2B^*} \left(\frac{E_F(e)}{m_e c^2}\right)^2 \right]^{\alpha+3/2} d\left(\frac{p_z}{m_e c}\right). \quad (17)$$

### 3.3. The Fermi Energy of Electrons in A Superhigh Magnetic Field

Since  $d(\frac{p_z}{m_e c}) = d(\frac{p_z c}{m_e c^2})$ , Eq.(17) can be further simplified

$$N_{pha} = \frac{2^{2(1-\alpha)}}{2\alpha+3} \frac{\pi}{(B^*)^{1+\alpha}} g_0 \left(\frac{m_e c}{h}\right)^3 \left(\frac{E_F(e)}{m_e c^2}\right)^{(2\alpha+4)}. \quad (18)$$

In order to modify the formula for  $E_F(e)$  in a SMF, let us refer to the Pauli exclusion principle (PEP) (see Ref. 24). According to the PEP, there are no two identical

fermions occupying the same quantum state simultaneously, thus highly degenerate electrons in a SMF have to fill quantum states from the lowest Landau level (the ground-state level) to the highest Landau level. In the mean time, according to quantum mechanics, the larger the electron Landau level number  $n$ , the more unstable the electron Landau level, and the electrons at a higher energy level are prone to have excited transitions towards a lower energy level by losing energy (e.g., releasing photons).

The electron Fermi energy, as the highest occupied state energy, corresponds to the maximum electron Fermi momentum. The resulted electron degeneracy pressure contributes a small fraction of the total dynamic pressure (mainly from neutron degeneracy pressure) countering against gravity collapse of a NS. Thus, the PEP not only explains higher energy levels of electrons, but also is responsible for the stability of the NS matter. In a NS, the electron number density  $n_e$  is determined by

$$n_e = N_A \rho Y_e, \quad (19)$$

where  $N_A = 6.02 \times 10^{23}$  is the Avogadro constant. According to PEP, the electron energy state number equals the electron number in a unit volume, we get

$$\begin{aligned} N_{pha} &= \frac{2^{2(1-\alpha)}}{2\alpha+3} \frac{\pi}{(B^*)^{1+\alpha}} g_0 \left(\frac{m_e c}{h}\right)^3 \\ &\times \left(\frac{E_F(e)}{m_e c^2}\right)^{2\alpha+4} = N_A \rho Y_e = n_e. \end{aligned} \quad (20)$$

By solving Eq.(20), we obtain

$$\begin{aligned} E_F(e) &= \left(\frac{2\alpha+3}{2^{(2-\alpha)\pi} g_0}\right)^{\frac{1}{2(\alpha+2)}} \times \\ &\left(\frac{h}{m_e c}\right)^{\frac{3}{2(\alpha+2)}} m_e c^2 (B^*)^{\frac{1+\alpha}{2(\alpha+2)}} n_e^{\frac{1}{2(\alpha+2)}}. \end{aligned} \quad (21)$$

Eq.(21) is a new general expression of  $E_F(e)$  in a SMF, where  $\frac{1+\alpha}{2(\alpha+2)}$  describes the magnetic field index of the expression. For the sake of convenience, the magnetic field index is denoted by a quantity  $\beta$ ,  $\beta = \frac{1+\alpha}{2(\alpha+2)}$ .

In order to discuss a reasonable range of  $\alpha$ , we generate a schematic diagram of  $\beta$  and  $\alpha$ . In Fig. 2, the dot-dashed line represents a singular point of  $\alpha = -2$ ; the dashed line represents  $\beta = 0$ , corresponding to  $\alpha = -1$ . The physical significance of  $\alpha = -1$  is that  $E_F(e)$  does not change with the variation of  $B$ , which equivalents to the case of weak-magnetic field approximation. The reasonable range of  $\alpha$  is thus estimated as  $\alpha < 0$  but  $\alpha \neq -1, -2$ . From Fig. 2, one can easily judge the relationship between  $E_F(e)$  and  $B$ : When  $\alpha < -2$  or  $-1 < \alpha < 0$ , the magnetic field index  $\beta > 0$ , and  $E_F(e)$  increases with  $B$ ; when  $-2 < \alpha < -1$ , the magnetic field index  $\beta < 0$ , and  $E_F(e)$  decreases with  $B$ .

Whether  $\beta$  is a positive or negative number depends on actual value of  $\alpha$ . From the relation of  $E_F(e)$  and  $n_e$  in a weak magnetic field approximation, it can be

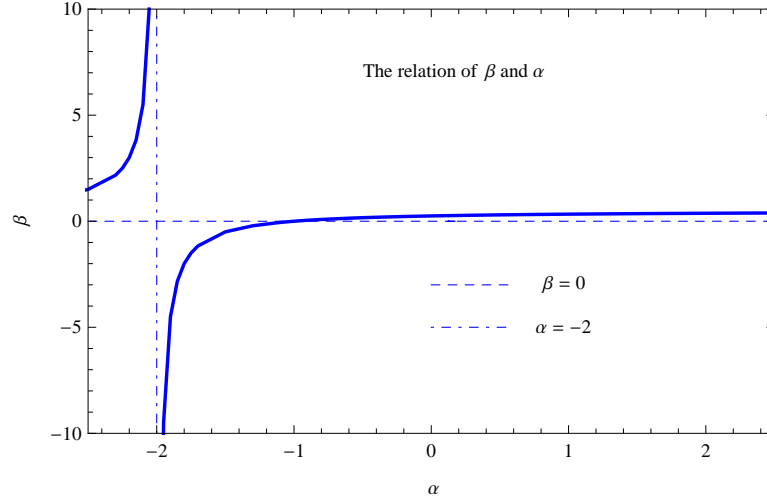


Fig. 2. The relation between  $\beta$  and  $\alpha$ .

seen that  $E_F(e)$  is solely determined by  $n_e$ , and  $E_F(e) \propto n_e^{1/3}$  (refer to Eq.(3) in Section 2). From the general expression of  $E_F(e)$  in a SMF (Eq.(20)), we can see that  $E_F(e) \propto (B^*)^{\frac{1+\alpha}{2(\alpha+2)}} n_e^{\frac{1}{2(\alpha+2)}}$ , i.e.,  $E_F(e)$  bears the relation not only to  $n_e$  but also to  $B$ , and the latter has great influence on the former by modifying electron phase space. It is also worth noticing that, the dimension of  $E_F(e)$  always remains unchanged no matter in a weak magnetic field or in a SMF, i.e.,  $E_F(e)$  is proportional to  $n_e^{1/3}$ . Thus, we obtain

$$\frac{1}{2(\alpha+2)} = \frac{1}{3}, \quad (22)$$

by comparing Eq.(3) with Eq.(21). Solving Eq.(22) yields  $\alpha = -0.5$ , which lies in a reasonable range of  $\alpha$ , as estimated above. Inserting  $\alpha = -0.5$  into Eq.(22), we get the magnetic field index  $\beta = 1/6$ . Compared with Eq.(1), the value of  $\beta$  obtained in this paper decreases by  $1/12$ . In spite of the minimum disparity, the modified magnetic field index is obviously superior to that of previous one (see Ref. 16, 17). This is because, in our previous works (see Ref. 16, 17), the magnetic field index  $\beta$  ( $\beta = 1/4$ ) corresponds to  $\alpha = 0$ , which means that different Landau levels have the same stability, and the differences between two different Landau levels are neglected.

In order to obtain an analytic expression for  $E_F(e)$  in a SMF, we assume that the ground state level has the highest stability, and the maximum value of  $g_n$  is  $g_0 = 1$ . This provides us with much convenience for theoretical derivations. Then the electron Landau-level stability coefficient (see Eq.(7)) can be expressed as  $g_1 = g_0 = 1$ , ( $n = 0, 1$ ) and  $g_n = n^{-\frac{1}{2}}$  ( $n \geq 2$ ).

Inserting  $\alpha = -0.5$  into Eq.(21), we have

$$\begin{aligned} E_F(e) &= 5.84 \times 10^{-11} (n_e')^{1/3} \\ &= 5.84 \times 10^{-11} (B^*)^{1/6} n_e^{1/3} \\ &= 5.84 \times 10^{-11} \left(\frac{B}{B_{cr}}\right)^{1/6} n_e^{1/3} \text{ (MeV)}, \end{aligned} \quad (23)$$

where  $n_e' = n_e (B^*)^{1+\alpha} = n_e (B^*)^{1/2}$  after considering the magnetic effects. Eq.(23) is a newly modified general expression for  $E_F(e)$  and  $n_e$  in a SMF.

By modifying the phase space of relativistic electrons, a SMF can enhance  $n_e$ , and decrease the maximum of electron Landau level number, resulting in a redistribution of electrons. As mentioned above, the strongly degenerate electrons have to occupy all possible microscopic states up to the highest Landau level, due to the requirement of the PEP. The enhanced  $n_e$  in a SMF means an increase in  $E_F(e)$ , corresponding to an increase in the electron degeneracy pressure.

If  $B$  is too high, e.g.,  $B > 10^{17}$  Gauss, the Landau cylinder will continuously elongate in the direction of  $B$ , and could become a narrow electron chain, then the maximum Landau level number is estimated to be  $n_m = 1$  or  $2$ . Under such circumstances, the premise of theoretical derivations in this paper will no longer apply. Thus the applicable conditions for Eq.(23) are limited by  $\rho \geq 10^7 \text{ g cm}^{-3}$  and  $B_{cr} \ll B \leq 10^{17}$  Gauss.

The Fermi energy of electrons in a magnetar is related to  $Y_e$ ,  $\rho$  and  $B$ . Inserting  $n_e = N_A Y_e \rho$  into Eq.(23), we obtain a new solution to  $E_F(e)$  in a SMF,

$$\begin{aligned} E_F(e) &= 59.1 \times \left(\frac{Y_e}{0.005647}\right)^{1/3} \left(\frac{\rho}{\rho_0}\right)^{1/3} (B^*)^{1/6} \\ &= 59.1 \times \left(\frac{Y_e}{0.005647}\right)^{1/3} \left(\frac{\rho}{\rho_0}\right)^{1/3} \left(\frac{B}{B_{cr}}\right)^{1/6} \text{ (MeV)}, \end{aligned} \quad (24)$$

where the relation of  $Y_e = Y_p \approx \frac{n_e}{n_n} \approx 0.005647 \times \left(\frac{\rho}{\rho_0}\right)$  (see Eq.(12) of Ref. 49) is used. We make schematic diagrams of  $E_F(e)$  and  $\rho$  in different magnetic fields, as shown in Fig. 3. In Fig. 3 the curves 1, 2 and 3 are fitted by Eq.(24), and the curve 4 is fitted by Eq.(2). When the matter density remains constant,  $\rho = \rho_0$ , we calculate the values of  $E_F(e)$  to be (102.41 – 162.58), (91.24 – 144.84) and (69.77 – 110.75) MeV, respectively when  $B^* = 100, 50$ , and  $10$ , respectively.

In order to compare the modified particular solution to  $E_F(e)$  with that in our previous works (see Ref. 17), we make schematic diagrams of  $E_F(e)$  and  $B$ , as shown in Fig.4.

When  $\rho = \rho_0$ , and  $B \sim (5.0 \times 10^{13} - 1.0 \times 10^{16})$  Gauss, we calculate the values of  $E_F(e)$  to be (60.34 – 145.92) MeV, and (46.1 – 134.83) MeV, respectively, by using Eq.(24) and Eq.(1), respectively. Fig. 4 clearly shows that the value of  $E_F(e)$  obtained with Eq.(24) is slightly bigger than that obtained with Eq.(2). This is mainly because that both the Fermi energy coefficient (59.1 MeV) and the density index (1/3) in Eq.(24) are higher than those in Eq.(1), although the magnetic field index  $\beta$  in Eq.(24) is lower than that in Eq.(2).

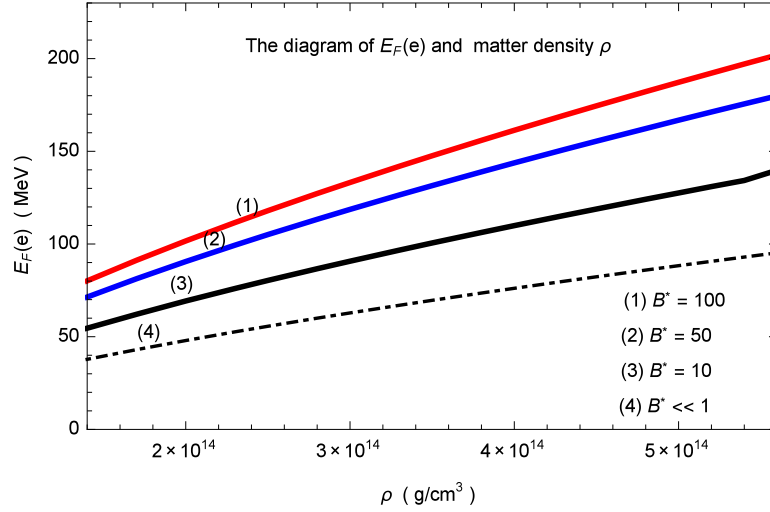


Fig. 3. The relations of  $E_F(e)$  and  $\rho$  in different magnetic fields. The matter density ranges  $\rho \sim (1.4 \times 10^{14} - 5.6 \times 10^{14}) \text{ g cm}^{-3}$ . The solid lines represent different strong magnetic fields and the dot-dashed line represents the weak magnetic field approximation ( $B^* \ll 1$ ).

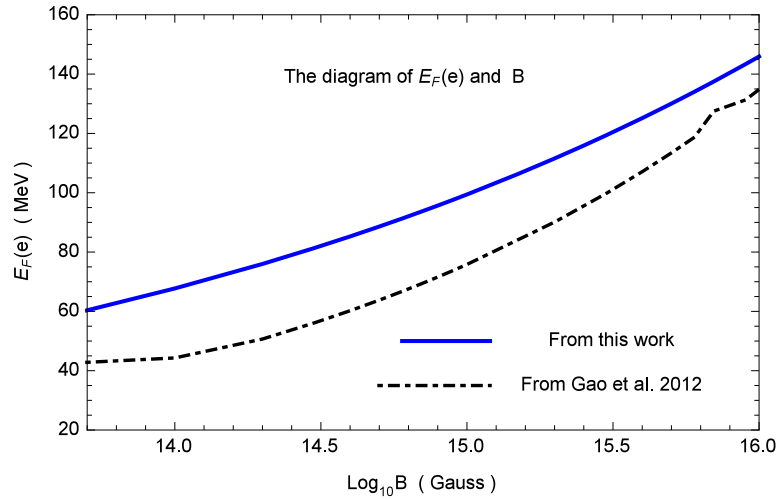


Fig. 4. The relations of  $E_F(e)$  and  $B$  in SMFs. The range of  $B$  is  $(5.0 \times 10^{13} - 1.0 \times 10^{16})$  Gauss. In order to facilitate the calculation, the matter density is assumed as  $\rho = \rho_0$ , arbitrarily. The solid line and the dot-dashed line are fitted by Eq.(24) and Eq.(1), respectively.

#### 4. Summary and Discussion

By introducing the stability coefficient of the electron Landau levels, we re-derive the general expression of  $E_F(e)$  in a SMF, and obtained a modified particular solution to

$E_F(e)$ . The solution has a lower magnetic field index of  $\beta = 1/6$ , which is lower than the previous one by a factor of  $1/12$ . Since there exists the discrepancy of stability in different Landau levels, we believe that this solution to  $E_F(e)$  is superior to the previous one (see Ref. 16, 17).

Just like in a weak magnetic field approximation, the value of the electron Fermi energy is determined solely by the electron number density. A SMF modifies the phase space of electrons, and thus increases the electron number density. The possible reasons are as follows:

(I) According to the requirement of electrical neutrality, the simple neutron decay and continuous electron capture occur simultaneously in the NS interior. However, a SMF will cause the former process to be faster than the latter (see Ref. 26). Since more neutrons in a SMF will be converted into protons, the proton fraction  $Y_p$  ( $Y_p = Y_e$ ) will increase, and the electron number density ( $n_e = N_A Y_e \rho$ ) will also increase correspondingly.

(II) According to nuclear physics, the proton fraction describes the asymmetry of nuclear matter, and the value of  $Y_p$  bears close relation to the symmetry energy, the symmetrical energy gradient, the incompressible coefficient, the volume bound energy and other parameters of nuclear matter (see Ref. 27, 28). A super-high magnetic field may increase the asymmetry of nuclear matter and improve the proton fraction. Thus, the average electron density of nuclear matter also increases correspondingly.

Although the structure, properties and EoS of nuclear matter are strongly influenced by a SMF, there has been no relevant and detailed research results in the physics community, the study on how a SMF influences the asymmetry (the influence on the proton abundance in particular) of nuclear matter will become one of our future research directions. Since the electron Fermi energy is one of most important parameters of EoS, a SMF will influence the EoS of a NS, as well as on the electron Fermi energy. Meanwhile, the particular solution to the electron Fermi energy obtained in this paper will surely influence calculations of the neutron-decay rates, the electron capture rates and the soft X-ray luminosity of a magnetar (see Ref. 16, 29, 17, 30).

As known to all, the cooling processes of a NS can be categorized into direct Urca and modified Urca reactions (see Ref. 31). Theoretical research shows that, direct Urca reactions in a ordinary NS with  $B \ll B_{\text{cr}}$  are strongly suppressed by Pauli blocking in a system composed of neutrons, protons and electrons, due to a high threshold (see Ref. 32, 33, 34, 35, 36) for the proton fraction  $Y_p \geq 1/9$ . Direct Urca reactions could take place (see Ref. 37, 18, 31) in the core of a supermassive NS, where  $Y_p$  could be higher than 0.11. However, when in a SMF, things could be quite different if we take into account of the magnetic effect on  $Y_p$ . This paper will be very useful in the future study on direct Urca reactions in the magnetar circumstances.

Modified Urca reactions are usually referred to as the standard cooling process

(see Ref. 31), in which the energy loss caused by resulting neutrinos plays an important role in the thermal evolution of a NS. Observations show that the surface thermal temperatures of most of high-magnetic field pulsars (see Ref. 38) are higher than those of ordinary radio pulsars with same characteristic ages. It is found that, the surface temperatures of magnetars (see Ref. 39, 40, 41, 44, 42 43) are also significantly higher than those of ordinary radio pulsars (see Ref. 44, 45). These observational findings not only challenge the standard cooling theory but also strongly hint that the magnetic field evolution and the thermal evolution of a magnetar are closely related. The results of this paper will facilitate the theoretical research on magneto-thermal evolution of magnetars, facilitate NS other properties research, such as superfluid, hyperons, asymmetry of nuclear matter and rotational evolution (see Ref. 46, 47, 48, 49).

It is expected that in the near future, our results will contribute to improving the standard model of neutron star cooling (see Ref. 31), and will be tested and developed by comparing the improved neutron star cooling model with magnetar spectrum observations

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## References

## References

1. C. Thompson and R. C. Duncan, *Astrophys. J.* **473** 322 (1996).
2. M. Colpi, U. Geppert and D. Page, *Astrophys. J. Lett.* **529** 29 (2000).
3. , P. M. Woods, C. Thompson, *Compact Stellar X-ray Sources*, (Edited by W. H. G Lewin and M. van der Klis, **547** 547 (2004).
4. Z. F. Gao, N. Wang, Q. H. Peng and et al., *Mod. Phys. Lett. A.* **28** 1350138 (2013).
5. Yi Hua. Lam, et al., *EPJ Web of Conferences* **66** 07011 (2014).
6. G. Martínez-Pinedo, Y H. Lam, K. Langanke., et al., *Phys. Rev. C.* **89** 045806 (2014).
7. X. H. Li, Z. F. Gao, X. D. Li and et al., *Int. J. Mod. Phys. D.* **25(1)** 1650002 (2016)
8. S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Drarfs, and Neutron Stars*, (New York, Wiley-Interscience, 1983)

9. G. Baym, C. Pethick and P. Sutherland, *Astrophys. J.* **170** 299 (1971).
10. S. B. Ruster, M. Hempel and J. Schaffnet-Bielich, *Phys. Rev. C.* **73** 035804 (2006).
11. S. Tsuruta, J. Sadino, A. akatsuka and et al., *Astrophys. J.* **691** 621 (2009)
12. U. Das and B. Mukhopadhyay, *Phys. Rev. D.* **86** 042001 (2012).
13. J. Dong, H. Zhang, L. Wang and W. Zuo, *Phys. Rev. C.* **87** 014302 (2013)
14. Z. F. Gao, N. Wang, Y. Xu, et al., *Astron. Nachr.* **336** No.8/9, 866 (2015)
15. E. J. Ferrer, V. de la. Incera, D. M. Paret and et al., *Phys. Rev. D.* **91** 085041 (2015).
16. Z. F. Gao, N. Wang, D.L. Song and et al., *Astrophys. Space Sci.* **334** 281 (2011).
17. Z. F. Gao, Q. H. Peng, N. Wang and et al., *Astrophys. Space Sci.* **342** 55 (2012).
18. D. Lai and S. L. Shapiro, *Astrophys. J.* **383** (1991) 745.
19. A. K. Harding and D. Lai, *Rep. Prog. Phys.* **69** (2006) 2631
20. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, ed. W. H. Freeman, (Pergamon Press, New York, 1965).
21. V. Canuto and J. Ventura, *Fund. Cosmic Phys.* **2** 203 (1977).
22. Q. H. Peng and H. Tong, *Mon. Not. R. Astron. Soc.* **378** 159 (2007).
23. R. Kubo, *Statistics Mechanics*, (Amsterdam: North-Holland Publishing Co. 1965)
24. W. Pauli, *Z. Physik*, **31**, 765 (1925).
25. R. K. Pathria, *Statistics Mechanics, 2nd*, (Singapore: Isevier. 2003).
26. S. Chakrabarty, D. Bandyopadhyay and S. Pal, *Phys. Rev. Lett.* **78** 75 (1997).
27. P. Wang and W. Zuo, *Phys. Rev. C.* **89** 054319 (2014).
28. P. Wang and W. Zuo, *Chin. Phys. C.* **39** 014101 (2015).
29. Z. F. Gao, N. Wang, D.L. Song and et al., *Astrophys. Space Sci.* **334** 281 (2011).
30. Z. F. Gao, Q. H. Peng, N. Wang and et al., *Chin. Phys. B.* **21** 057109 (2012).
31. D. G. Yakovlev, A. D. Kaminker, O. Y. Gnedin and et al. *Phys. Rep.* **354** 1 (2001).
32. J. Boguta and A. R. Bodmer *Nuclear. Physics. A.* **292** 413 (1977).
33. J. Boguta. *Phys. Lett.* **106** 255 (1981).
34. J. Boguta and H. Stocker *Physics. Lett.* **120** 289 (1983).
35. K. Takahashi, *Journal of Physics G-Nuclear and Particle Physics*, **35(3)** 035002 (2008).
36. E. M. Henley, M. B. Johnson and L. S. Kisslinger, *Phys. Rev. D.* **76** 125007 (2007).
37. D. A. Baiko and D. G. Yakovlev, *Astron. Astrophys.* **342** 192 (1999).
38. H. Zhu, W. W. Tian and P. Zuo, *Astrophys. J.* **793** 95 (2014).
39. A. D. Kaminker, D. G. Yakovlev, A. Y. Potekhin and et al., *Mon. Not. R. Astron. Soc.* **371** 477 (2006).
40. S. Mereghetti, *Astron. Astrophys. Rev.* **15** 225 (2008).
41. A. D. Kaminker, A. A. Kaurov, A. Y. Potekhin and D. G. Yakovlev, *Mon. Not. R. Astron. Soc.* **442** 3484 (2014).
42. Z. F. Gao, X. D. Li, N. Wang, et al., *Mon. Not. R. Astron. Soc.* **456** 55 (2016).
43. Jing-Jing Liu, *Research in Astronomy and Astrophysics*. (2016) arXiv:1602.05501
44. S. B. Popov, *Publications of the Astronomical Society of Australia* **30** 045 (2013).
45. S. B. Popov, A. A. Kaurov and A. D. Kaminker, *Publications of the Astronomical Society of Australia* **32** 018 (2015)
46. W.H. Long, et al., *Phys. ReV. C.*, **85** 025806 (2012)
47. Z. H. Li and W. Zuo, *Phys. ReV. C.*, **85** 037001 (2012)
48. B. Qi, et al., *Chin. Phys. Lett.* **32** 112101 (2015)
49. A. Li, et al., *Phys. ReV. C.*, **93** 015803 (2016)